Oxford Cambridge and RSA

AS Level Further Mathematics A

Y531/01 Pure Core

Monday 14 May 2018 – Afternoon

Time allowed: 1 hour 15 minutes

You must have:
- Printed Answer Booklet
- Formulae AS Level Further Mathematics A

You may use:
- a scientific or graphical calculator

MODEL SOLUTIONS

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Answer **all** the questions.

1 **(i)** Find a vector which is perpendicular to both $\begin{pmatrix}1\\3\\-2\end{pmatrix}$ and $\begin{pmatrix}-3\\-6\\4\end{pmatrix}$. **[2]**

1i) Using the cross product

$$\begin{pmatrix}1\\3\\-2\end{pmatrix} \times \begin{pmatrix}-3\\-6\\4\end{pmatrix} = \begin{pmatrix}i & j & k\\1 & 3 & -2\\-3 & -6 & 4\end{pmatrix} = \begin{pmatrix}12-12\\-(4-6)\\-6+9\end{pmatrix}$$

$$= \begin{pmatrix}0\\2\\3\end{pmatrix}$$

OR

Using simultaneous equations

$$\begin{pmatrix}1\\3\\-2\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$x + 3y - 2z = 0$$

$$\begin{pmatrix}-3\\-6\\4\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$-3x - 6y + 4z = 0$$

let $y = 2$

$x + 3(2) - 2z = 0$

$x - 2z = -6$ — ①

$-3x - 6(2) + 4z = 0$

$-3x + 4z = 12$ — ②

$2 \times$ ① $+$ ②

$$\begin{aligned} 2x - 4z &= -12 \\ + \quad -3x + 4z &= 12 \\ \hline -x &= 0 \\ x &= 0 \end{aligned}$$

∴ when $x=0$, $y=2$, sub into $x+3y-2z=0$

$0 + 3(2) - 2z = 0$

$6 = 2z$

$3 = z$

∴ vector $= \begin{pmatrix}0\\2\\3\end{pmatrix}$

(ii) The cartesian equation of a line is $\frac{x}{2} = y-3 = 2z+4$.

Express the equation of this line in vector form. **[3]**

$$\frac{x}{2} = y-3 = 2z+4$$

$$\frac{x-0}{2} = \frac{y-3}{1} = \frac{z+2}{\frac{1}{2}}$$

$$\underline{r} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

2 **In this question you must show detailed reasoning.**

The cubic equation $2x^3 + 3x^2 - 5x + 4 = 0$ has roots α, β and γ. By making an appropriate substitution, or otherwise, find a cubic equation with integer coefficients whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. **[3]**

let $u = \frac{1}{x} \Rightarrow \frac{1}{u} = x$

$$2\left(\frac{1}{u}\right)^3 + 3\left(\frac{1}{u}\right)^2 - 5\left(\frac{1}{u}\right) + 4 = 0$$

$$\frac{2}{u^3} + \frac{3}{u^2} - \frac{5}{u} + 4 = 0$$

$\times u^3$

$$2 + 3u - 5u^2 + 4u^3 = 0$$

$$\underline{\underline{4u^3 - 5u^2 + 3u + 2 = 0}}$$

<u>OR</u>

$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{3}{2}$

$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} = -\frac{5}{2}$

$\alpha\beta\gamma = \frac{-d}{a} = -\frac{4}{2} = -2$

$\therefore$ as roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma} = \frac{-\frac{5}{2}}{-2} = \frac{5}{4}$$

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma} = \frac{-\frac{3}{2}}{-2} = \frac{3}{4}$$

$$\frac{1}{\alpha\beta\gamma} = -\frac{1}{2}$$

$\therefore u^3 - \frac{5}{4}u^2 + \frac{3}{4}u + \frac{1}{2} = 0$

$\Rightarrow \underline{\underline{4u^3 - 5u^2 + 3u + 2 = 0}}$

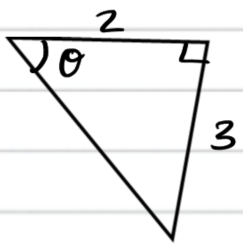
3 **In this question you must show detailed reasoning.**

The complex numbers z_1 and z_2 are given by $z_1 = 2 - 3\text{i}$ and $z_2 = a + 4\text{i}$ where a is a real number.

(i) Express z_1 in modulus-argument form, giving the modulus in exact form and the argument correct to 3 significant figures. **[3]**

$|z_1| = \sqrt{2^2 + (-3)^2}$

$= \sqrt{13}$



$\arg z_1 = \theta$

$\arctan\left(\frac{-3}{2}\right) = -0.983$ (3sf)

$z_1 = \sqrt{13}\left(\cos(-0.983) + i\sin(-0.983)\right)$

$z_1 = \sqrt{13}\left(\cos(0.983) - i\sin(0.983)\right)$

(ii) Find $z_1 z_2$ in terms of a, writing your answer in the form $c + \text{i}d$. **[2]**

$(2-3i)(a+4i) = 2a - 3ai + 8i - 12i^2$

$= (2a+12) + i(8-3a)$

(iii) The real and imaginary parts of a complex number on an Argand diagram are x and y respectively. Given that the point representing $z_1 z_2$ lies on the line $y = x$, find the value of a. **[2]**

$y = x$

$8 - 3a = 2a + 12$

$-4 = 5a$

$a = -\frac{4}{5}$

(iv) Given instead that $z_1 z_2 = (z_1 z_2)^*$ find the value of a. **[2]**

Imaginary part = 0

$8 - 3a = 0$

$8 = 3a$

$a = \frac{8}{3}$

4 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 2 & a \end{pmatrix}$.

(i) Show that $\det\mathbf{A} = 6 - 3a$. **[2]**

$$\det A = 2(-a-2) - (a-2) + 2(2+2)$$
$$= -2a - 4 - a + 2 + 8$$
$$= -3a + 6$$
$$= 6 - 3a$$

(ii) State the value of a for which **A** is singular. **[1]**

$$\det A = 0$$
$$6 - 3a = 0$$
$$6 = 3a$$
$$\underline{2 = a}$$

(iii) Given that **A** is non-singular find $\mathbf{A}^{-1}$ in terms of a. **[4]**

Matrice of cofactors $= \begin{pmatrix} -a-2 & 2-a & 4 \\ 4-a & 2a-4 & -2 \\ 3 & 0 & -3 \end{pmatrix}$

Transpose $= \begin{pmatrix} -a-2 & 4-a & 3 \\ 2-a & 2a-4 & 0 \\ 4 & -2 & -3 \end{pmatrix}$

$$A^{-1} = \frac{1}{6-3a} \begin{pmatrix} -a-2 & 4-a & 3 \\ 2-a & 2a-4 & 0 \\ 4 & -2 & -3 \end{pmatrix}$$

5 In this question you must show detailed reasoning.

(i) Express $(2+3i)^3$ in the form $a + ib$. **[3]**

$(2+3i)^3 = 2^3 + 3(2)^2(3i) + 3(2)(3i)^2 + (3i)^3$
$= 8 + 36i + 54i^2 + 27i^3$
$= 8 + 36i - 54 - 27i$
$= -46 + 9i$

(ii) Hence verify that $2+3i$ is a root of the equation $3z^3 - 8z^2 + 23z + 52 = 0$. **[3]**

$3(2+3i)^3 - 8(2+3i)^2 + 23(2+3i) + 52$
$= 3(-46+9i) - 8(4+12i-9) + 46 + 69i + 52$
$= -138 + 27i - 8(12i-5) + 98 + 69i$
$= -40 + 96i - 96i + 40$
$= 0$
Hence $2+3i$ is a root.

(iii) Express $3z^3 - 8z^2 + 23z + 52$ as the product of a linear factor and a quadratic factor with real coefficients. **[4]**

If $2+3i$ is a root, then $2-3i$ is also a root as it is its conjugate pair.
This means $(z-(2+3i))$ and $(z-(2-3i))$ are factors.

$(z-(2+3i))(z-(2-3i))$
$= z^2 - 2z + 3zi - 2z + 4 - 6i - 3zi + 6i - 9i^2$
$= z^2 - 4z + 4 + 9$
$= z^2 - 4z + 13$

$(z^2 - 4z + 13)(az + b)$
$= az^3 + bz^2 - 4az^2 - 4bz + 13az + 13b$
compare coefficients
z^3 coeff. : $a = 3$
z^2 coeff. : $b - 4a = -8$
$b - 4(3) = -8$
$b - 12 = -8$
$b = 4$

$\therefore (z^2 - 4z + 13)(3z + 4)$

OR

$$\begin{array}{r} 3z+4 \\ z^2-4z+13 \overline{)\,3z^3-8z^2+23z+52} \\ -\;3z^3-12z^2+39z \\ \hline +4z^2-16z+52 \\ -\;4z^2-16z+52 \\ \hline 0 \end{array}$$

$$\therefore (z^2-4z+13)(3z+4)$$

6 The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} t & 6 \\ t & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2t & 4 \\ t & -2 \end{pmatrix}$ where t is a constant.

(i) Show that $|\mathbf{A}| = |\mathbf{B}|$. [2]

$|A| = -2t - 6t = -8t$

$|B| = -4t - 4t = -8t$

$\therefore |A| = |B|$

(ii) Verify that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$. [3]

$$AB = \begin{pmatrix} t & 6 \\ t & -2 \end{pmatrix}\begin{pmatrix} 2t & 4 \\ t & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2t^2+6t & 4t-12 \\ 2t^2-2t & 4t+4 \end{pmatrix}$$

$|A||B| = 8t \times 8t$
$= 64t^2$

As $64t^2 = 64t^2$
then $|A||B| = |AB|$

$|AB| = (2t^2+6t)(4t+4) - (2t^2-2t)(4t-12)$
$= 8t^3+24t^2+8t + 24t - 8t^3 + 24t^2 + 8t^2 - 24t$
$= 64t^2$

(iii) Given that $|\mathbf{AB}| = -1$ explain what this means about the constant t. [2]

$64t^2 = -1$

$t^2 = -\frac{1}{64}$

$t = \pm \frac{i}{8}$

$\therefore t$ is imaginery.

7 Prove by induction that $2^{n+1}+5\times9^n$ is divisible by 7 for all integers $n\geqslant1$. **[6]**

when $n=1$

$2^{1+1}+5\times9^1 = 2^2+5\times9$

$= 4+45$

$=49$

$=7\times7$

As 49 is divisible by 7, then when $n=1$, it is divisible by 7.

Assume $n=k$ is divisible by 7.

$f(k)=2^{k+1}+5\times9^k$ is divisible by 7.

When $n=k+1$

$2^{k+1+1}+5\times9^{k+1}$

$f(k+1) = 2\times2^{k+1}+5\times9\times9^k$

$= 2\times2^{k+1}+7\times5\times9^k+2\times5\times9^k$

$= 2(2^{k+1}+5\times9^k)+7\times5\times9^k$

$= 2f(k)+7\times5\times9^k$

So true for $n=k$ $\therefore$ true for $n=k+1$. But true for $n=1$. So true for all positive integers $n\geqslant1$.

8 The 2×2 matrix **A** represents a transformation T which has the following properties.

- The image of the point $(0,1)$ is the point $(3,4)$.
- An object shape whose area is 7 is transformed to an image shape whose area is 35.
- T has a line of invariant points.

(i) Find a possible matrix for **A**. **[8]**

let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$

$b=3$, $d=4$.

The area has been scaled by 5, meaning $\det A=5$

$ad-bc=5$

$4a-3c=5$ — 1

Invariant points: $\begin{pmatrix} a & 3 \\ c & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$ax + 3y = x$ $\qquad$ $cx + 4y = y$

$x(1-a) = 3y$ $\qquad$ $3y = -cx$

Set these two equations equal to eachother:

$x(1-a) = -cx$

$1-a = -c$

$1+c = a$

Sub this into equation ①

$4(1+c) - 3c = 5$

$4 + 4c - 3c = 5$

$4 + c = 5$

$c = 1$

$\therefore a = 1+1 = 2$

$\therefore \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

The transformation S is represented by the matrix **B** where **B** $= \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

(ii) Find the equation of the line of invariant points of S. **[2]**

$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$3x + y = x$ $\qquad$ $2x + 2y = y$

$y = -2x$ $\qquad$ $y = -2x$

$\therefore$ invariant line is $y = -2x$

(iii) Show that any line of the form $y = x + c$ is an invariant line of S. **[3]**

$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ x+c \end{pmatrix} = \begin{pmatrix} 3x + x + c \\ 2x + 2x + 2c \end{pmatrix}$

$= \begin{pmatrix} 4x + c \\ 4x + 2c \end{pmatrix}$

This is also in the form $y = x + c$

$4x + 2c = (4x + c) + c$

$4x + 2c = 4x + 2c.$

Hence the line $y = x + c$ is invariant under S for all c.